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The generating analytic element approach with application to the modified Helmholtz equation

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Abstract In this paper a new method for obtaining functions with a given singular behavior that satisfy a class of partial differential equations is presented. Differential equations of this class contain operators of the form ∇^{2n}, where n is a positive integer. The method uses Wirtinger calculus which enables one to invert the Laplacian in combination with the decomposition method introduced by Adomian at the end of the twentieth century. The procedure uses a singular holomorphic function as its basis, and constructs the solution term by term as an infinite series of functions; the process consists of an infinite number of steps of integration. This method is applied to construct a number of singular solutions to the modified Helmholtz equation in the context of groundwater flow. These functions are discharge potentials, which are two-dimensional functions by definition. The gradient of the discharge potential is the vertically integrated flow over the thickness of an aquifer, or water-bearing layer. The discharge potentials of interest here are those used in the analytic element method. This method, as originally conceived, relies on the superposition of suitably chosen holomorphic functions, and is a form of a method known as the Trefftz method, not to be confused with the Trefftz method applied to finite element techniques. The main analytic elements used are singular line elements, characterized by either a jump along the element in the tangential or the normal component of the discharge vector. The analytic line elements for the case of divergence-free irrotational flow are well established and many of these are forms of singular Cauchy integrals. Application of the analytic element method to more general cases of flow, governed for example by the modified Helmholtz equation (flow in systems of aquifers separated by leaky layers) and the heat equation (transient flow) is possible using the method presented in this paper. The latter application is beyond the scope of this paper, but it is worth noting that for that case the constant that occurs in the modified Helmholtz is replaced by a general function of time and application of Laplace transforms can be avoided. A method for constructing such functions is presented; the procedure for constructing these functions is referred to as the generating analytic element approach. Application of this approach requires the existence of the holomorphic singular line element. The approach is discussed and an example for the case of a line-sink for a system of two aquifers separated by a leaky layer and bounded above by in impermeable boundary is presented.

Keywords Analytic element method · Groundwater flow · Modified Helmholtz equation · Superposition of solutions · Wirtinger calculus

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1 Introduction

Our objective is to present a new method for obtaining functions with a given singular behavior at either a point, along a curve, or along a straight line. These functions satisfy a member of a certain class of partial differential equations whose leading terms contain an operator of the form ∇^{2n}, where n is a positive integer. They are expressed in terms of an infinite series of functions that are created by successive application of the inverse Laplacian, ∇^{-2}. The first function in this series is always either the real or the imaginary part of a holomorphic function with the desired singular behavior. The method relies on the use of Wirtinger calculus [1], [2, pp. 63–70] in combination with the decomposition method [3, pp. 6–337].

The singular functions thus created are useful in the analytic element method [4, pp. 404–514], [5, pp. 205–231], [6], and the special issues devoted to this technique: Journal of Hydrology, 1999, Volume 226, and Ground Water, 2004, Volume 43(6), 2005, Volume 44(1). We refer to the method to be presented in this paper as the generating analytic element approach, abbreviated as GAEA, and introduce it by creating analytic elements suitable for modeling groundwater flow in systems of leaky aquifers. Flow in such systems is governed by the modified Helmholtz equation.

The analytic element method is a technique based upon the idea that boundary-value problems can be solved by superposition of suitably chosen base functions (which makes it a Trefftz method) which each contain certain degrees of freedom. In the analytic element method we are ultimately concerned with determining a vector field, for the case of this paper a two-dimensional one. Nearly all of the base functions model discontinuities of either the normal or the tangential component of the vector field along a linear boundary segment; the element simulates this discontinuity. The base functions are then superimposed and their degrees of freedom determined so as to model linear two-sided boundaries, boundaries of sub-domains, or boundaries that separate parts of the domain from other parts.

The analytic element method differs from most related techniques by one or more of the following properties: the freedom of choice in how the analytic element is constructed, the stand-alone character of each analytic element in that it satisfies well-defined conditions that can be adjusted as needed by variation of its free parameters, and in that the analytic elements, always represented by functions, are defined throughout the infinite domain. Discontinuities in the parameters that enter into the partial differential equations, such as aquifer properties in the case of groundwater flow, do not require a subdivision of the domain; the appropriate analytic elements simulate the required discontinuities across the boundaries of such sub-domains and can be evaluated on either side of these discontinuities; the line-doublets presented in this paper were derived for this purpose, but their use will not be demonstrated in this paper.

The analytic element method, AEM, has been applied successfully to a range of groundwater flow problems. We refer the reader to [6] for an overview of this method. Extension of the AEM to problems governed by the modified Helmholtz equation has been achieved, but so far with moderate success. Bakker and Strack [7] developed line elements of constant strength (first order) which suffer from the drawback that the length of the element is limited by the leakage factor, a factor that depends solely on the properties of the system.

Leaky aquifer systems consist of water-bearing layers (aquifers) separated by layers of relatively low hydraulic conductivity. Solutions to flow in systems of leaky aquifers require the solution of a system of coupled partial differential equations; see e.g., [8, pp. 377–395]. Solutions for cases with wells in such systems are well known; see [7–11], and consist of a sum of solutions to the modified Helmholtz equation. The example presented in this paper is concerned with the case that the upper boundary is impermeable. The potential defined in each of the aquifers consists of the sum of a single harmonic function and a solution to the modified Helmholtz equation. The latter solutions are Bessel functions for the case of a well that is screened in any of the aquifers. It may be noted that the method of solution presented here can be readily extended to the case of multiple aquifers using the methodology introduced by [9].

The functions that constitute the discharge potential for leaky aquifer flow are either harmonic or pan harmonic; pan-harmonic functions satisfy the modified Helmholtz equation. The pan-harmonic potential for a line-sink was first obtained by integration along a line of a polynomial representation of the Bessel function by Heitzman [12], and was later used by Keil [13] to obtain an approximate expression for flow in a system of leaky aquifers with an unconfined upper boundary. The potential presented by Heitzman is valid only in a relatively small region near the line element, and the extraction rate along the line-sink is a constant. Bakker and Strack [7] presented an expression for a line-sink that was obtained also by integration of a polynomial expression of the Bessel function that has a wider range of validity than that used by Heitzman, but also has a limited range of validity and has a constant rate of extraction along the element. Although this function can be evaluated at much larger distances from the element than the one developed by Heitzman, it still suffers from the same basic problem of a limited range of applicability, which forces subdivision of the object to be modeled in segments such that the solution remains applicable over the segment. As a consequence, the subdivision of the elements depends on the aquifer data, and thus limits the flexibility of a model based on such elements.

We present an approach in this paper and apply it to obtain functions that satisfy the modified Helmholtz equation with any desired accuracy. These functions are useful i-n analytic element models of regional groundwater flow. Such models involve very large numbers of elements (thousands) and computational efficiency is a major issue. A study carried out by Hanson [14] shows that leakage induced by a well in a single leaky aquifer can be neglected for all practical purposes beyond distances of six times the leakage factor Λ, i.e., a factor that depends solely on the aquifer properties. We propose to disregard leakage induced by the element considered, at distances beyond eight times the leakage factor. For the case of radial flow toward a point sink in an infinite aquifer, described mathematically by the Bessel function $\mathrm{K}_0\,(r/\Lambda)$, this implies that its values can be neglected for $r \geq 8\Lambda$. In order to obtain a consistent solution, i.e., a solution that satisfies a given differential equation and boundary conditions, we will construct a solution to the modified Helmholtz equation that satisfies the condition that the potential vanishes along a circle of radius 8Λ. The effect of the approximation can then be assessed by computing the total discharge over the bounding circle, which should be sufficiently small to be neglected for the application in question. It is important to note that this approach is not necessary; it is possible to apply the approach and yet obtain a solution that is valid throughout the domain, but is expected to require far more computational effort. We present the solution for this case of radial flow toward a point sink, and will discuss the magnitude of the difference between the Bessel function and the function presented here. Note, however, that the discrepancy between the two functions is due only to the difference in boundary conditions, and does not imply that the new solution does not satisfy the modified Helmholtz equation.

2 GAEA: The generating analytic element approach

We demonstrate the approach first for some elementary cases, for which solutions already are in existence. This has the advantage that the results are verifiable by comparison with the existing functions, and has the further advantage of relative simplicity.

The approach is based upon first selecting a holomorphic function that has the desired singular behavior; the singularity may be at infinity. Examples that we will use are the Taylor series (singular only at infinity), the function that has a logarithmic singularity (a well) at the origin, and, finally the potentials for the line-sink and the line-doublet, or the single and double layers, respectively. The chosen holomorphic function is used as the generating analytic element in the GAEA to construct a function that satisfies the modified Helmholtz equation. It is worth remembering, however, that the generating analytic element approach is not restricted to the Helmholtz equation, but can be applied equally well to obtain solutions to other differential equations.

In what follows we demonstrate how the GAEA makes it possible, by a process of successive integration, to construct basic analytic elements that are solutions of the modified Helmholtz equation and can then be superimposed to obtain solutions to boundary value problems. This solution consists of an infinite series of functions; convergence of this series is determined on a case-by-case basis by considering the form of the solution. We will focus our attention first on the modified Helmholtz equation by itself, and later apply these solutions to solve some problems of leaky aquifer flow.

We write the modified Helmholtz equation in the following form

$$\nabla^2\Phi = \frac{\Phi}{\Lambda^2}, \tag{1}$$

where Λ [L] is the leakage factor; we will express this parameter with the dimension of length in terms of physical properties when discussing the leaky aquifer system.

We will make use of the complex variable $z = x + \mathrm{i}y$ defined in the physical plane, its complex conjugate $\bar{z} = x - \mathrm{i}y$, and Wirtinger calculus, which is based on the reversible coordinate transformation

$$z = x + \mathrm{i}y \quad \bar{z} = x - \mathrm{i}y, \tag{2}$$

$$x = \frac{1}{2}(z + \bar{z}) \quad y = \frac{1}{2\mathrm{i}}(z - \bar{z}). \tag{3}$$

Using the standard rules that govern coordinate transformations, we can transform the components of vectors and tensors, in particular the Laplacian, which transforms as

$$\nabla^2\Phi = 4\frac{\partial^2\Phi}{\partial z\partial\bar{z}}. \tag{4}$$

We will concern ourselves mainly with line segments; the case of the Taylor series and that of the point sink are exceptions. We introduce a dimensionless complex variable Z for the former cases as

$$Z = \frac{2z - (z_1 + z_2)}{z_2 - z_1}, \tag{5}$$

where z_1 and z_2 represent the coordinates of the end points of a line in the $z = x + \mathrm{i}y$ plane, so that (4) may be transformed

$$\nabla^2\Phi = 4\frac{\partial^2\Phi}{\partial z\partial\bar{z}} = 4\frac{\partial^2\Phi}{\partial Z\partial\bar{Z}}\frac{dZ}{dz}\frac{d\bar{Z}}{\bar{d}z} = \frac{16}{L^2}\frac{\partial^2\Phi}{\partial Z\partial\bar{Z}} = \frac{\Phi}{\Lambda^2}, \tag{6}$$

where

$$L^2 = (z_2 - z_1)(\bar{z}_2 - \bar{z}_1). \tag{7}$$

We see from (6) that

$$\frac{\partial^2\Phi}{\partial Z\partial\bar{Z}} = \beta^2\Phi, \tag{8}$$

where β is a dimensionless constant,

$$\beta = \frac{L}{4\Lambda}, \tag{9}$$

We introduce a series of real functions $\underset{n}{H}$ with the following property

$$\frac{\partial^2 \underset{n}{H}}{\partial Z\partial\bar{Z}} = \underset{n-1}{H} \tag{10}$$

and select the first function $\underset{0}{H}$ as the generating analytic element for the GAEA, which we represent as $\underset{0}{\Phi}$, i.e,

$$\underset{0}{H} = \underset{0}{\Phi}, \tag{11}$$

where $\underset{0}{H}$ is harmonic, i.e.,

$$\frac{\partial^2 \underset{0}{H}}{\partial Z\partial\bar{Z}} = 0. \tag{12}$$

With the function $\underset{0}{H}$ known, $\underset{1}{H}$ can be found by successive integration of (10) with respect to Z and $\bar{Z}$ for $n = 1$. Repeating this procedure will yield expressions for the functions $\underset{n}{H}$. It is important to note that this integration process is possible thanks to the representation (4) of the Laplacian.

We discuss irrotational groundwater flow as the application in this paper, and refer to the function to be constructed in that context. The vertically integrated discharge in an aquifer is represented in complex form as

$$W = Q_x - \mathrm{i}Q_y = -2\frac{\partial \Phi}{\partial z}, \tag{13}$$

where Q_x and Q_y are the components of the discharge vector, and Φ is called the discharge potential. We write this discharge potential as the following infinite series

$$\Phi = \sum_{n=0}^{\infty} \beta^{2n} \underset{n}{H}. \tag{14}$$

Substitution of this expression for Φ in the differential equation (8) yields

$$\sum_{n=1}^{\infty} \beta^{2n} \underset{n-1}{H} = \beta^2 \sum_{n=0}^{\infty} \beta^{2n} \underset{n}{H}. \tag{15}$$

We renumber indices in the first sum and consider only a finite number of terms, N,

$$\sum_{n=0}^{N-1} \beta^{2(n+1)} \underset{n}{H} = \beta^2 \sum_{n=0}^{N} \beta^{2n} \underset{n}{H}. \tag{16}$$

The difference between the left and the right side of the equation is given by

$$\frac{\partial^2 \Phi}{\partial Z \partial \bar{Z}} - \beta^2 \Phi = \beta^{2n} \underset{N}{H}. \tag{17}$$

If the series of functions converges, then, given an infinite number of terms, the differential equation will be satisfied exactly. The series of functions presented in this paper all converge absolutely, and we will be able to reduce the error due to truncation to within machine accuracy. Thus, the differential equation is met exactly in the same manner as we can compute the Bessel function exactly; we are limited by machine precision.

The procedure can be carried out in terms of complex functions; this is often an advantage because the generating analytic element is known as a function of a single complex variable. In that case, we use complex functions $\underset{n}{\Xi}$ to replace the real functions $\underset{n}{H}$ so that

$$\frac{\partial^2 \underset{n}{\Xi}}{\partial Z \partial \bar{Z}} = \underset{n-1}{\Xi}. \tag{18}$$

The potential is the real part of the sum of functions $\underset{n}{\Xi}$, each multiplied by a term β^{2n}

$$\Phi = \tfrac{1}{2} \sum_{n=0}^{N} \beta^{2n} \left[\underset{n}{\Xi} + \overline{\underset{n}{\Xi}} \right]. \tag{19}$$

3 The non-singular holomorphic function

If the generating analytic element does not contain any singularities in the domain of interest, then we can represent the function $\underset{n}{\Xi}$ in general form as follows. Consider the following expression

$$\underset{n}{\Xi} = \frac{\bar{Z}^n}{n!} \underset{n}{f}(Z), \tag{20}$$

where the functions $\underset{n}{f}(Z)$ have the following property

$$\frac{\mathrm{d} \underset{n}{f}}{\mathrm{d}Z} = \underset{n-1}{f}. \tag{21}$$

This function has the desired property (18). We verify this by first differentiating (20) with respect to $\bar{Z}$

$$\frac{\partial \underset{n}{\Xi}}{\partial \bar{Z}} = \frac{n\bar{Z}^{n-1}}{n!}\underset{n}{f}(Z) \tag{22}$$

and next differentiate with respect to Z, which yields

$$\frac{\partial^2 \underset{n}{\Xi}}{\partial Z \partial \bar{Z}} = \frac{\bar{Z}^{n-1}}{(n-1)!}\underset{n-1}{f}(Z) = \underset{n-1}{\Xi}. \tag{23}$$

Indeed, the function $\underset{n}{\Xi}$ has the desired property (18).

We mentioned in passing that this procedure is not applicable for singular line elements. This is so because such functions exhibit discontinuities along a line connecting their singular points to infinity. Since these discontinuities are multiplied by powers of $\bar{Z}$, undesired singular behavior results. We need to introduce a different form of the function for singular generating analytic elements. The exception is the point sink, where it appears fortuitously that the potential does not have a branch cut at all.

It is important to note that the procedure outlined above produces a solution to the differential equation that has desired singular behavior.

4 The Taylor series

We use the GAEA first to generate a function Φ that satisfies (1) in the finite domain with no singularities inside the circle that encloses the domain. We do this by choosing as the generating function the Taylor series:

$$\underset{0}{\Xi} = \sum_{m=0}^{\infty} \underset{0}{\Xi}_m = \sum_{m=0}^{\infty} a_m Z^m. \tag{24}$$

We consider a single power Z^m, and construct the function $\underset{n}{f}$ according to the definition (21) which yields, denoting the power by the index m

$$\underset{n}{f}_m(Z) = \frac{Z^{m+n}}{(m+n)!}. \tag{25}$$

We apply this to each term in the Taylor series, use (20), and obtain

$$\underset{n}{\Xi} = \sum_{m=0}^{\infty} a_m \left\{ \frac{\bar{Z}^n Z^{m+n}}{n!(m+n)!} \right\} = \sum_{m=0}^{\infty} a_m \left\{ \frac{(\bar{Z}Z)^n Z^m}{(m+n)!n!} \right\}. \tag{26}$$

An expression for the potential Φ is obtained by the use of (19)

$$\Phi = \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\beta^{2n}}{(m+n)!n!}\left[a_m \bar{Z}^n Z^{m+n} + \bar{a}_m Z^n \bar{Z}^{m+n}\right]. \tag{27}$$

Note that each term contains a factor $1/[(m+n)!n!]$ which will cause the individual terms to decrease in value rapidly beyond a certain number of terms, causing the series to converge to some fixed value. This appearance of factorials turns out to occur in all the applications discussed in this paper. Note that the series converges absolutely; the ratio of the modulus of the $(n+1)$st term divided by that of the nth one vanishes for $n \to \infty$.

The truncation error due to taking a finite number of terms in this case consists of two parts; the first part is due to truncation of the Taylor series that represents the generating analytic element; the second part is due to truncation of the series of functions.

5 The point sink

We choose the complex potential for the point sink of unit discharge as the next generating analytic element. We carry out this analysis for a complex function $\underset{0}{\Xi}$, defined as

$$\underset{0}{\Xi} = \frac{1}{2\pi} \log Z, \quad -\pi < \Im\mathfrak{m} \log Z \leq \pi, \tag{28}$$

where we take the branch cut, in this and all further uses of the complex logarithm, along the negative real axis; Z is a dimensionless complex variable, defined for this case as

$$Z = z/R, \tag{29}$$

so that the differential equation in dimensionless form becomes for this case

$$\nabla^2 \Phi = 4\frac{\partial^2 \Phi}{\partial z \partial \bar{z}} = \frac{4}{R^2}\frac{\partial^2 \Phi}{\partial Z \partial \bar{Z}} = \frac{\Phi}{\Lambda^2}, \tag{30}$$

where R is an arbitrary length. We observe that

$$\frac{\partial^2 \Phi}{\partial Z \partial \bar{Z}} = \beta^2 \Phi, \tag{31}$$

where

$$\beta = \frac{R}{2\Lambda}. \tag{32}$$

We introduce a function $\underset{n}{f}$ defined as

$$\underset{n}{f}(Z) = \frac{1}{n!} Z^n \left[\log Z - \psi^*_{n+1}\right], \quad n \geq 1, \tag{33}$$

where

$$\psi^*_{n+1} = \sum_{m=1}^{n} \frac{1}{m}, \quad n \geq 1,$$
$$\psi^*_{n+1} = 0, \quad n = 0. \tag{34}$$

This function has the property that

$$\frac{\mathrm{d}\underset{n}{f}}{\mathrm{d}Z} = \underset{n-1}{f} \tag{35}$$

and we see from (28) that

$$\underset{0}{\Xi} = \frac{1}{2\pi} \underset{0}{f}. \tag{36}$$

We use (20) to obtain an expression for $\underset{n}{H}$ which yields

$$\underset{n}{H} = \frac{1}{2}\left[\underset{n}{\Xi} + \underset{n}{\bar{\Xi}}\right] = \frac{1}{4\pi n!}\left[\bar{Z}^n \underset{n}{f} + Z^n \underset{n}{\bar{f}}\right]. \tag{37}$$

We may write this with (33) in the form

$$\underset{n}{H} = \frac{1}{4\pi n!}\left[\frac{Z^n \bar{Z}^n}{n!} \log Z + \frac{\bar{Z}^n Z^n}{n!} \log \bar{Z} - 2\frac{Z^n \bar{Z}^n}{n!} \psi^*_n\right] \tag{38}$$

or

$$\underset{n}{H} = \frac{1}{4\pi (n!)^2} (Z\bar{Z})^n \left[\log[Z\bar{Z}] - 2\psi^*_n\right]. \tag{39}$$

This potential may be expressed in terms of a radial coordinate r, with its dimensionless form r/R and we have

$$\frac{r}{R} = \sqrt{Z\bar{Z}}. \tag{40}$$

The potential for a point sink of unit discharge becomes, using (19):

$$\Phi = \sum_{n=0}^{\infty} \beta^{2n} \underset{n}{H}(Z, \bar{Z}). \tag{41}$$

6 Solving boundary-value problems

The approach to solving boundary-value problems using the analytic element method consists of creating singular solutions to the partial differential equation considered. The singular solutions have degrees of freedom, embodied in the function that controls the singularity. These singular functions usually represent a discontinuity of a magnitude given by a series expansion, such as a polynomial or a Fourier series. The singular solutions are chosen in such a way that their contribution to the far-field behavior of the solution is controlled, i.e., the boundary condition at infinity appropriate to the element in question is applied to each function individually. A characteristic of the analytic element method is that each solution to the governing equations can be constructed in isolation; superposition is used to obtain the solution to the desired problem; only then are the elements chosen according to the phenomenon that they have to represent, and the degrees of freedom are used to ensure that the sum of individual solutions satisfies the boundary conditions with the desired accuracy.

The general solution to the differential equation considered in the preceding section consists of a linear combination of two linearly independent solutions. The solution for the case of a point sink in an infinite domain consists of a term that is singular at the origin, the modified Bessel function of the second kind and order zero, $\mathrm{K}_0(r/\Lambda)$, which represents the radial flow toward the point sink and approaches minus the natural logarithm near the point sink. The second solution is the modified Bessel function of the first kind and order zero, $\mathrm{I}_0(r/\Lambda)$, which is equal to 1 with a derivative of zero at the origin, and becomes infinite for $r \to \infty$. The generating analytic element approach is based upon the idea of generating a solution to the partial differential equation, using the Adomian decomposition method combined with Wirtinger calculus. This approach will yield the singular term plus a non-singular solution; for the case of the point sink, this implies that the solution will also contain some unknown factor times the modified Bessel function I_0. The modified Bessel function K_0 (r/Λ) decreases rapidly with increasing values of r/Λ, and is usually neglected in the field of applied groundwater flow at distances larger than about 4Λ. Hanson [14], showed that, for the case of a well in a multi-aquifer system, the effect of the well (point sink) can be neglected at distances over 6Λ. A difficulty in computing the modified Bessel function $\mathrm{K}_0(r/\Lambda)$ for large r/Λ is that many terms in the expansion need to be taken into account in order to compute the very small values of the function. In this respect its behavior is similar to that of the expansion of the function e^{-x} about $x = 0$.

We have two options to determine the non-singular solution uniquely, and to deal with the accuracy problems at large distances. The first, and perhaps the most elegant from a mathematical viewpoint, is to use the solution obtained using GAEA and couple it to the general solution to the modified Helmholtz equation valid between a circle of finite radius and one of a radius that approaches infinity, which can be written in terms of an infinite series of products of solutions to the separated equation in two dimensions; see [15, p. 16]. This can be done using the principle of holomorphic matching [16], which makes it possible to link the two solutions along a chosen interzonal boundary in such a way that both components of the gradient of the potential are continuous across that boundary.

We will use an alternative approach here in view of numerical efficiency. We neglect the effect of the source of influence at distances larger than 8Λ by adding a non-singular solution and requiring that the singular and non-singular solutions vanish at a circle centered at the singularity and of radius 8Λ. Comparison with the exact solution for the case of the point sink shows that the approach renders a solution that approximates the exact one very well.

We can determine a second linearly independent solution to the modified Bessel equation by taking a second generating analytic element, valid for this case of radial flow, besides the singular one, $\log r/\Lambda$. The only other function that satisfies Laplace's equation for the special case of radial flow is, besides the logarithm, the constant; this leads to the series (27) with the generating analytic element the constant a_0, i.e., all $a_m = 0$, $m > 0$. We obtain, representing this potential as $\underset{e}{\Phi}$

$$\underset{e}{\Phi} = \frac{a_0}{4\pi} \sum_{n=0}^{\infty} \frac{\beta^{2n}}{(n!)^2} (Z\bar{Z})^n. \tag{42}$$

The complete solution to the problem consists of the sum of the series for the point sink and the latter series

$$\Phi = \frac{a_0}{4\pi} \sum_{n=0}^{\infty} \frac{\beta^{2n}}{(n!)^2} (Z\bar{Z})^n + \sum_{n=0}^{\infty} \beta^{2n} \underset{n}{H}(Z, \bar{Z}). \tag{43}$$

We use expression (39) for $\underset{n}{H}$ and obtain

$$\Phi = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{\beta^{2n}}{(n!)^2} (Z\bar{Z})^n \left\{ \log[Z\bar{Z}] - 2\psi_n^* + a_0 \right\}. \tag{44}$$

We set $z\bar{z}$ equal to $R^2 = (\kappa\Lambda)^2$, where κ is a positive number, and set the potential equal to zero, which yields, noticing that $Z\bar{Z} = 1$ for $z\bar{z} = R^2$

$$\Phi = \frac{1}{4\pi} \sum_{n=0}^{\infty} \left[\frac{\beta^n}{n!} \right]^2 \left[-2\psi_n^* + a_0 \right] = 0. \tag{45}$$

We solve this for a_0 and obtain

$$a_0 = 2 \frac{\sum_{n=0}^{\infty} (\beta^n/n!)^2 \psi_n^*}{\sum_{n=0}^{\infty} (\beta^n/n!)^2}, \tag{46}$$

where

$$\beta = \frac{R}{2\Lambda} = \frac{\kappa}{2}. \tag{47}$$

One of the reviewers of this paper noted that it can be shown, using equations (9.6.12) and (9.6.13) in [17, p. 375], that this solution is identical to the solution to the stated boundary-value problem using the Bessel functions K_0 and I_0

$$\Phi = -\frac{1}{2\pi} \left[\mathrm{K}_0(r/\Lambda) - \frac{\mathrm{K}_0(R/\Lambda)}{\mathrm{I}_0(R/\Lambda)} \mathrm{I}_0(r/\Lambda) \right]. \tag{48}$$

The difference between the modified boundary-value problem with a boundary with fixed potential at given R/Λ and the solution represented by the single Bessel function $\mathrm{K}_0(r/\Lambda)$ may be well expressed in terms of the relative difference in discharge that crosses the boundary $r = R$ for the two cases. We express this relative difference by the use of (48) as follows

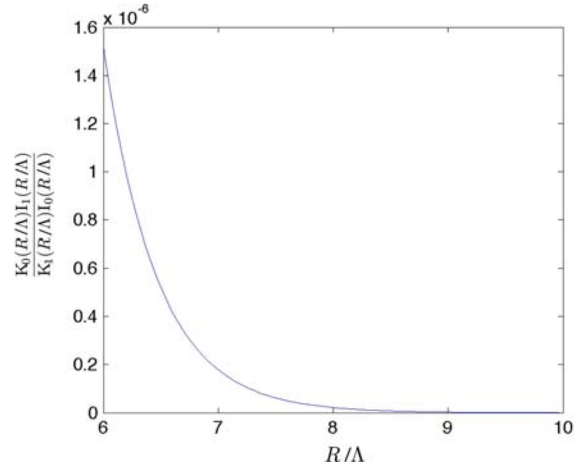
$$D = \frac{\mathrm{K}_0(R/\Lambda)\mathrm{I}_1(R/\Lambda)}{\mathrm{K}_1(R/\Lambda)\mathrm{I}_0(R/\Lambda)}. \tag{49}$$

This quantity is plotted versus R/Λ in Fig. 1.

7 The general case

We will determine the individual functions for the singular line elements in a similar fashion. As for the case of the point sink, the complete function will consist of two parts: the singular line element itself, determined by the GAEA, and a non-singular solution. The boundary condition that the potential vanishes along a fixed boundary, in this case an ellipse, rather than a circle, will be used to express the unknown coefficients in the non-singular

Fig. 1 Plot of the relative difference in discharge flowing over the boundary $r = R$ as a function of R/Λ



solution, the Taylor series, in terms of the unknown constants that control the discontinuity of the line element. The functions presented in the following sections will contain an arbitrary solution to the governing equation as will become clear from the derivation. In fact, we will add specially chosen functions to ensure that the discontinuity is limited to the element, rather than extends to infinity. One such function is added at each successive integration of the Laplacian as part of the process of constructing the solution. The solution will become unique only after adding the function obtained using the Taylor series as the generating analytic function.

Our purpose is to create individual analytic elements, to be used when constructing models by superposition. We therefore are concerned only with the properties of these special functions and, in particular, the convergence of their infinite series. The interested reader is referred to the literature for a more general treatment of the Adomian decomposition method [3].

8 Line elements

The functions that we presented so far in this paper are reformulations of existing solutions. The functions that we present in what follows, however, are new. The singular line elements that we refer to here as line-doublets are also called potentials for double layers, and are obtained from a singular Cauchy integral, see [4, pp. 283–302] and [18]. The latter author introduced the method of over-specification, which makes it possible to achieve a high degree of accuracy for the holomorphic line elements of high order in terms of boundary values. We derive in this section the potentials for a line sink and a line doublet that satisfy the modified Helmholtz equation. We use the GAEA, with the high order line-elements as the generating analytic elements. The derivation of the expressions for the potentials for the line elements require special attention; these elements exhibit branch cuts and we must take care not to introduce unwanted discontinuities. We develop a technique for constructing these functions in such a way that only the generating analytic element is singular and exhibits a discontinuity; all the higher order terms in the series, i.e., all terms for $n \geq 1$, will not exhibit a jump.

We choose the points z_1 and z_2 in expression (5) for the dimensionless variable Z as the endpoints of the element, and L represent its length. The differential equation has the form (8) with β given by (9).

$$\frac{\partial^2 \Phi}{\partial Z \partial \bar{Z}} = \beta^2 \Phi. \tag{50}$$

Note that in each of the successive integrations of (50) with respect to Z and $\bar{Z}$, an arbitrary real function of the form $h(Z) + \bar{h}(\bar{Z})$ can be added to the solution. We choose such additional functions in such a way that only the first term in the expansion, the generating analytic element, has a discontinuity. However, other functions could be added. The addition of a non-singular solution, obtained with the Taylor series as the generating analytic element, will make the final solution unique, after the application of the boundary condition along the bounding ellipse as we did for the case of the point sink.

9 The line doublet

9.1 The holomorphic functions F and f

The generating analytic element used for the line-doublet is discussed in [4, p. 297]. The line-doublet has the property that the potential is discontinuous across the element, and therefore the tangential component of its gradient also, except when the jump in potential is a constant. Line-doublets are useful for modeling discontinuities in the tangential component of flow, such as they occur along a boundary that separates two domains with different transmissivities. The complex potential $\underset{\text{db}}{\Omega}$ is the holomorphic function with the discharge potential as its real part and the stream function as its imaginary part. The stream function is continuous across the line-doublet; a discontinuity in the stream function corresponds to a discontinuity in the normal component of flow, i.e., to extraction along the element. It is an essential property of the line-doublet that only the tangential component of flow jumps, whereas the normal component of flow is continuous. The complex potential $\underset{\text{db}}{\Omega}$ represents a line doublet of order M, and has the form

$$\underset{\text{db}}{\Omega} = \frac{1}{2\pi \mathrm{i}} \sum_{m=0}^{M} a_m Z^m \log \frac{Z-1}{Z+1} + \frac{1}{2\pi \mathrm{i}} \sum_{m=0}^{M-1} a_m P_m(Z) \tag{51}$$

where the a_m are real coefficients to be determined from the boundary condition along the element,

$$\mathfrak{Im} a_m = 0, \quad m = 0, 1, \ldots M. \tag{52}$$

The function $P_m(Z)$ is a polynomial of degree $m-1$ with coefficients determined from the condition that $\underset{\text{db}}{\Omega}$ is of order $1/Z$ for $Z \to \infty$,

$$P_m(Z) = \sum_{j=0}^{m-1} \underset{m}{\beta_j} Z^j, \quad m = 0, 1, \ldots M. \tag{53}$$

For the present analysis we consider only the term $\underset{\text{db}}{\Omega}_m$ defined as

$$\underset{\text{db}}{\Omega}_m = \frac{Z^m}{2\pi \mathrm{i}} \log \frac{Z-1}{Z+1} + \frac{1}{2\pi \mathrm{i}} P_m(Z), \quad m = 0, 1, \ldots M. \tag{54}$$

We write

$$Z^m = \{(Z-1)+1\}^m = \sum_{k=0}^{m} \binom{m}{k} (Z-1)^k \tag{55}$$

and

$$Z^m = \{(Z+1)-1\}^m = \sum_{k=0}^{m} \binom{m}{k} (Z+1)^k (-1)^{m-k}, \tag{56}$$

where

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}. \tag{57}$$

We may now write (54) as follows

$$\underset{\text{db}}{\Omega}_m = \frac{1}{2\pi \mathrm{i}} \left\{ \sum_{k=0}^{m} \alpha_{mk} \left[(Z-1)^k \log(Z-1) - (-1)^{m-k} (Z+1)^k \log(Z+1) \right] + P_m(Z) \right\}. \tag{58}$$

We introduce functions $\underset{n}{F}_m$ as follows

$$\underset{n}{F}_m(Z) = \frac{1}{\mathrm{i}}\left\{\sum_{k=0}^{m}\alpha_{mk}\left[\underset{n}{f}_k(Z-1) - (-1)^{m-k}\underset{n}{f}_k(Z+1)\right] + \underset{n}{P}_m(Z)\right\},$$
$$m = 0, 1, \ldots M, \quad n = 0, 1 \ldots, \tag{59}$$

where

$$\alpha_{mk} = \binom{m}{k} \tag{60}$$

$$\underset{n}{f}_m(Z) = \frac{m!}{(n+m)!}Z^{m+n}\left[\log Z - \sum_{j=1}^{n}\frac{1}{m+j}\right], \quad n \geq 1 \tag{61}$$

$$\underset{n}{f}_m(Z) = \frac{m!}{(n+m)!}Z^{m+n}\log Z \quad n = 0, \tag{62}$$

and

$$\underset{n}{P}_m(Z) = \sum_{j=0}^{m-1}\underset{m}{\beta}_j\,\underset{n}{p}_j(Z), \quad (n \geq 1), \quad \underset{0}{P}_m(Z) = P_m(Z). \tag{63}$$

The functions $\underset{n}{p}_m$ are defined as

$$\underset{n}{p}_j(Z) = \frac{Z^{j+n}}{(j+1)(j+2)\ldots(j+n)}, \tag{64}$$

so that

$$\frac{\mathrm{d}\underset{n}{p}_j(Z)}{\mathrm{d}Z} = \underset{n-1}{p}_j(Z), \quad \underset{n}{p}_j(Z) = \underset{n-1}{p}_j(Z)\frac{Z}{j+n}. \tag{65}$$

We observe that (58) can be expressed in terms of $\underset{0}{F}_m$ as follows

$$\underset{\mathrm{db}}{\Omega}_m = \frac{1}{2\pi}\underset{0}{F}_m(Z). \tag{66}$$

The functions $\underset{n}{F}_m$ and $\underset{n}{f}_m$ have the property that

$$\frac{\mathrm{d}\underset{n}{F}_m}{\mathrm{d}Z} = \underset{n-1}{F}_m, \quad \frac{\mathrm{d}\underset{n}{f}_m}{\mathrm{d}Z} = \underset{n-1}{f}_m. \tag{67}$$

We may now write the line doublet that is used as the generating analytic element as the following sum

$$\Omega_{\mathrm{db}} = \sum_{m=0}^{M}\frac{a_m}{2\pi}\underset{0}{F}_m(Z). \tag{68}$$

9.2 The potential Φ

We write the potential Φ for the line doublet as a sum of functions as follows

$$\Phi = \frac{1}{2\pi}\sum_{m=0}^{M}a_m\Re\mathfrak{e}\,\underset{0}{F}_m(Z) + \frac{1}{4\pi}\sum_{m=0}^{M}a_m\sum_{n=1}^{\infty}\beta^{2n}\left[\underset{n}{\Xi}_m(Z,\bar{Z}) + \overline{\underset{n}{\Xi}_m(Z,\bar{Z})}\right], \tag{69}$$

where M is the degree of the polynomial of the line doublet that serves as the generating analytic function and where the a_m are real coefficients to be determined from the boundary conditions along the element. We recall that the basis function for the non-singular generating analytic element was chosen as follows, see (20)

$$\underset{n}{\Xi} = \frac{\bar{Z}^n}{n!}\underset{n}{f}(Z). \tag{70}$$

This representation fails here because of the presence of a branch cut. We modify (70) in such a way that all terms for $n \geq 1$ are multiplied by a factor $\bar{Z} - Z$, which vanishes along $Z = \bar{Z}$, i.e, along the element, thus eliminating the effect of the branch cut. It appears that we can meet condition (18) only if we resort to a sum of functions to represent each term and write, with m again representing the power of Z

$$\underset{n}{\Xi}_m = \sum_{j=0}^{n-1} \frac{\underset{n}{B}_j}{(n-j)!}(\bar{Z} - Z)^{n-j} \underset{n+j}{F}_m(Z), \quad m = 0, 1, \ldots M, \quad n \geq 0, \tag{71}$$

where the constants $\underset{n}{B}_j$ are real and independent of the power m

$$\mathfrak{Im}\underset{n}{B}_j = 0, \quad n = 0, 1, \ldots, \quad j = 0, 1, \ldots n \tag{72}$$

and where

$$\underset{n}{B}_{-1} = 0. \tag{73}$$

It it demonstrated in Appendix A that the function $\underset{n}{\Xi}_m$ indeed meets condition (18), provided that the constants $\underset{n}{B}_j$ meet the following constraints

$$\underset{n}{B}_j = \underset{n}{B}_{j-1} + \underset{n-1}{B}_j, \quad j = 0, 1, 2, \ldots n-1, \quad n = 1, 2, \ldots \tag{74}$$

with the choice

$$\underset{0}{B}_0 = 1. \tag{75}$$

The recursive relation (74) together with (75) makes it possible to determine all of the coefficients involved in the expression for the potential, except the coefficients a_m which are to be determined from the boundary conditions along the element. Note that the coefficients $\underset{n}{B}_j$ can be computed a priori; they are independent both of the order m and the parameter β.

9.3 The discharge function

The discharge function is defined as minus two times the complex derivative of the potential with respect to z,

$$W = -2\frac{\partial \Phi}{\partial z} = -2\frac{\partial \Phi}{\partial Z}\frac{\partial Z}{\partial z} = -2\frac{\partial \Phi}{\partial Z}\frac{2}{L}\mathrm{e}^{-\mathrm{i}\alpha}, \tag{76}$$

where L is the length of the line element and α the angle between the line element and the x axis. Note that $\partial\Phi/\partial z$ is not equal to the derivative of the function $\underset{n}{\Xi}_m$, given by (71). The potential is given by (69),

$$\Phi = \frac{1}{2\pi}\sum_{m=0}^{M} a_m \mathfrak{Re}\underset{0}{F}_m(Z) + \frac{1}{4\pi}\sum_{m=0}^{M} a_m \sum_{n=1}^{\infty} \beta^{2n} \underset{n}{H}_m(Z), \tag{77}$$

where the function $\underset{n}{H}_m(Z)$ is defined as

$$\underset{n}{H}_m(Z) = \sum_{j=0}^{n-1} \frac{\underset{n}{B}_j}{(n-j)!}(\bar{Z} - Z)^{n-j}\left[\underset{n+j}{F}_m(Z) - (-1)^{n+j}\underset{n+j}{F}_m(\bar{Z})\right],$$
$$m = 0, 1, \ldots M, \quad n = 0, 1, \ldots, \tag{78}$$

where we used that $(-1)^{n-j} = (-1)^{n+j}$ and made use of the definition of $\underset{n+j}{F}{}_m$, which implies that

$$\overline{\underset{n+j}{F}{}_m(Z)} = -\underset{n+j}{F}{}_m(\bar{Z}). \tag{79}$$

Differentiation of a term $\underset{n}{H}{}_m$ with respect to Z gives

$$\begin{aligned}\frac{\partial \underset{n}{H}{}_m}{\partial Z} &= \sum_{j=0}^{n-1} \frac{-\underset{n}{B}{}_j}{(n-j-1)!}(\bar{Z}-Z)^{n-j-1}\left[\underset{n+j}{F}{}_m(Z) - (-1)^{n+j}\underset{n+j}{F}{}_m(\bar{Z})\right] \\ &\quad + \sum_{j=0}^{n-1} \frac{\underset{n}{B}{}_j}{(n-j)!}(\bar{Z}-Z)^{n-j}\underset{n+j-1}{F}{}_m(Z)\end{aligned} \tag{80}$$

The derivative $\partial\Phi/\partial Z$ is obtained from (77) and (80)

$$\frac{\partial\Phi}{\partial Z} = \frac{1}{4\pi}\sum_{m=0}^{M} a_m \frac{\mathrm{d}\underset{0}{F}{}_m(Z)}{\mathrm{d}Z} + \frac{1}{4\pi}\sum_{m=0}^{M} a_m \sum_{n=1}^{\infty} \frac{\beta^{2n}\partial \underset{n}{H}{}_m(Z)}{\partial Z}. \tag{81}$$

10 The line-sink

10.1 The holomorphic function G

The generating analytic function for the next case is the complex potential for a line-sink of high order, also known as the complex potential for the single layer. The real part of this complex function is the potential and the imaginary part is the stream function. The normal component of flow is discontinuous across the element, whereas the tangential component is continuous; this property corresponds to a continuous potential and a discontinuous stream function. It is important to note that the stream function has meaning only for the generating analytic element; it is not single valued for divergent flow. We introduce the function $\underset{n}{G}{}_m$ as follows

$$\underset{n}{G}{}_m(Z) = \sum_{k=1}^{m} \alpha_{mk}\left\{\underset{n}{f}{}_k(Z-1) - (-1)^{m-k}\underset{n}{f}{}_k(Z+1)\right\} + \underset{n}{P}{}_m(Z), \quad n \geq 0. \tag{82}$$

The functions $\underset{n}{G}{}_m$ differ from the functions $\underset{n}{F}{}_m$ in two ways only. First, the latter contains a factor $1/i$ outside the summation sign, whereas the former does not. Second, the sum extends from zero to m for the functions $\underset{n}{F}{}_m$, whereas it extends from one to m for the case of $\underset{n}{G}{}_m$. The result of these differences is that the line-sink has a continuous component of the gradient of the potential along the element, whereas its normal component jumps across the element. Furthermore, the potential of the line-doublet is infinite at the end points of the element, whereas the potential of the line-sink is finite at these points.

The functions $\underset{n}{G}{}_m$ and $\underset{n}{f}{}_m$ have the property that

$$\frac{\partial \underset{n}{G}{}_m}{\partial Z} = \underset{n-1}{G}{}_m, \quad \frac{\partial \underset{n}{f}{}_m}{\partial Z} = \underset{n-1}{f}{}_m, \tag{83}$$

(compare (59), (61) and (62)). The line-sink that is the generating analytic function is

$$\underset{\mathrm{ls}}{\Omega} = \sum_{m=0}^{M} \frac{b_m}{2\pi}\underset{0}{G}{}_m, \tag{84}$$

where the b_m are real constants, to be determined from the boundary conditions

$$\mathfrak{Im}\, b_m = 0, \quad m = 0, 1, \ldots M. \tag{85}$$

10.2 The potential Φ

We write the potential Φ for the line-sink as an infinite sum of functions as follows

$$\Phi = -\frac{1}{4\pi}\sum_{m=0}^{M} b_m \sum_{n=0}^{\infty} \beta^{2n} \underset{n}{H}_m(Z, \bar{Z}), \tag{86}$$

where the minus sign is used so that a positive strength corresponds to extraction, where M is the degree of the polynomial of the line-sink that serves as the generating analytic function. The functions $\underset{n}{H}_m$ are given by

$$\underset{n}{H}_m = \sum_{j=0}^{n} \frac{\underset{n}{C}_j}{(n-j)!}(\bar{Z}-Z)^{n-j}\left[\underset{n+j}{G}_m + (-1)^{n-j}\underset{n+j}{\bar{G}}_m\right] \quad m = 0, 1, \dots M, \quad n \geq 0 \tag{87}$$

where the constants $\underset{n}{C}_j$ are real and independent of m

$$\Im\mathfrak{m}\underset{n}{C}_j = 0, \quad n = 0, 1, 2, \dots, \quad j = 0, 1, 2, \dots. \tag{88}$$

We observe from (86) and (87) that the constant $\underset{0}{C}_0$ is equal to 1:

$$\underset{0}{C}_0 = 1. \tag{89}$$

In Appendix B it is shown that

$$\underset{n+j}{G}_m = \underset{n+j}{G}_m(Z), \quad m = 0, 1, \dots M, \quad n, j = 0, 1, 2, \dots, \tag{90}$$

$$\underset{n+j}{\bar{G}}_m = \underset{n+j}{G}_m(\bar{Z}), \quad m = 0, 1, \dots M, \quad n, j = 0, 1, 2, \dots. \tag{91}$$

Note that the function G is real for real values of Z.

The form (87) is similar to that chosen for the line-doublet, but not quite the same. The sum now runs for j all the way up to n, so that not all functions $\underset{n}{H}_m$ are multiplied by a factor $\bar{Z} - Z$. The reason for this choice is that the potential for the line-sink is continuous across the element, as opposed to that for the line-doublet. The generating analytic element for the line-sink has a discontinuous stream function, rather than a discontinuous potential. Because the stream function does not exist for the higher-order terms we must enforce the condition of continuity directly to the discharge vector for all terms with $n \geq 1$. We demonstrate in Appendix B that the form (87) indeed renders a potential function that satisfies the conditions that the Laplacian of the next term equals the previous term in the series of functions, and that the continuity condition is satisfied.

11 Boundary conditions

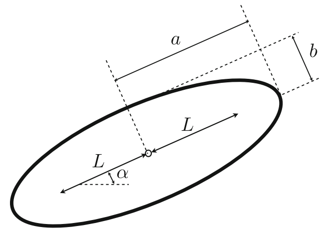
We stated, when discussing the potential for a well, that the solution obtained by applying GAEA leads to a combination of the singular solution with a non-singular solution. A boundary condition other than that along the element is required to determine the coefficients in the non-singular solution (27). Once again, rather than applying the boundary condition at infinity, we apply the condition along a boundary far enough from the element that the function can be neglected beyond that boundary. We choose for the boundary an ellipse that has the end points of the line element as its foci, and require that the potential, which is the sum of the singular and non-singular solutions vanishes along that boundary; see fig. 2.

We map the exterior of the ellipse onto the dimensionless χ plane, shown in Fig. 3, using the transformation

$$Z = X + \mathrm{i}Y = \tfrac{1}{2}\left(\frac{\chi}{\nu} + \frac{\nu}{\chi}\right), \tag{92}$$

so that the ellipse corresponds to $\chi\bar{\chi} = 1$. We let χ take on values on the circle and require that the sum of the potential due to the line element and the potential (27) cancel. Note that inversion of this transformation is not required in this case. These conditions are used to generate a set of linear equations in terms of unknown $2N$ complex (a_m) and one real (a_0) coefficients, where N is the number of terms included in the series.

Fig. 2 The line element and bounding ellipse



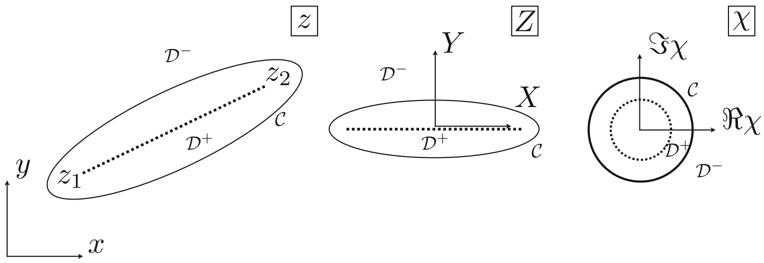


Fig. 3 The z, Z, and χ planes

12 Numerical considerations

Implementation of the procedure outlined in this paper in a computer program requires considerable care in how the various functions and terms are computed. Although the series of functions presented in this paper converge absolutely, this does not imply that the terms can always be computed with sufficient accuracy to lead to meaningful results. For example, evaluation of the effect of high-order elements (e.g., line elements with polynomials of degree 10 or higher), along the bounding ellipse lead to considerable inaccuracies if programmed directly as presented above. It is well established for high-order elements, e.g., [4, pp. 291–298], [18,19] that asymptotic expansions are needed to compute the complex potentials accurately at distances away from the element; usually, a switch to the asymptotic expansion is made outside of a circle that is only slightly larger than half the element length. The same is true for the series of holomorphic functions used here. Using asymptotic expansions, computation is possible at sufficiently large distance from the element; recall that the functions must be computed along the bounding ellipse.

We expand all holomorphic functions, such as $\underset{0}{F}_m$ and $\underset{0}{G}_m$, asymptotically in terms of a Laurent series of exclusively negative terms, then integrate each term with respect to Z to obtain $\underset{n}{F}_m$ and $\underset{n}{G}_m$. At each step a constant of integration must be added; we compute these constants by requiring that the expanded functions are equal to the original ones. The corresponding analysis is straightforward, but is needed for numerical implementation. We present the derivation of the expansions in Appendix C.

13 Applications

We present applications in the context of groundwater flow. However, the applications are not chosen for their hydrological relevance, but in order to test the method thoroughly, and to demonstrate that it can deal with problems that put the technique under a great deal of strain. For this reason, the governing parameters are chosen so that the effects are concentrated near the elements, and that strong singularities occur at the end of line elements and at the points were the elements meet. We achieve this by enforcing a constant value of the discharge potential along the elements. In practice, piezometric heads, and therefore the discharge potentials, tend to vary along such features as narrow streams, rather than be constant, but the objective of the paper justifies the choice of values that do not commonly occur in practice. An additional benefit of enforcing constant values along the elements is that it can be seen by visual inspection of contour plots that the potential and head are indeed constant along the elements. For

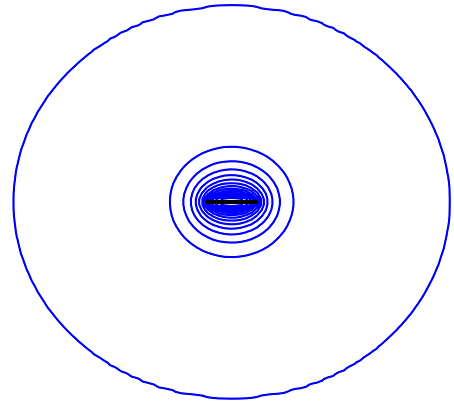


Fig. 4 Contours of constant leakage for $\Lambda = \frac{1}{2}L$

this same reason, the values of potentials and heads are, in themselves, irrelevant and are omitted from the plots. The shape of the contours and the fact that they do not intersect the line elements is the real test of the accuracy of the solution, besides the numerical verification in computer models that the partial differential equations are indeed satisfied, along with the boundary conditions. However, conditions of variable piezometric head along the boundaries, as well as other conditions such as those along streams with leaky bottoms (bottoms with resistance to flow, e.g., resulting from stream deposits) can be applied. The computer program in which these functions are currently implemented (called MLAEM) is capable of simulating such conditions.

The expressions for the potentials for the line-sink and the line-doublet are implemented in routines written in object-oriented FORTRAN 95. We present computational results obtained with the computer program MLAEM for a few cases of groundwater flow below. The first case concerns a single semi-confined aquifer, i.e., an aquifer of constant thickness separated from a domain (a lake, for example) of constant head by a leaky layer. The second case concerns a system of two aquifers, confined above by an impermeable upper boundary, and separated by a leaky layer. We use line-sinks as the elements that induce leakage in both cases.

13.1 A single line-sink in a semi-confined aquifer

The first case concerns flow in a semi-confined aquifer. For that case the solution consists of a solution to the modified Helmholtz equation with the parameter Λ given by the equation

$$\Lambda^2 = cT, \tag{93}$$

where c [L] is the resistance of the leaky layer, and T the transmissivity of the aquifer. We present results for a line-sink of constant strength for two cases. The first example is valid for $\Lambda = \frac{1}{2}L$ and the second one corresponds to $\Lambda = (1/10)L$, i.e., one-twentieth of the element length. The contour levels are shown in Fig. 4 for the first example and in Fig. 5 for the second one. The contour levels are not of interest by themselves; the difference in value in potential for adjacent contours differ by a constant amount, of course, and the contour plots clearly illustrate that there is virtually no flow near the bounding ellipse. The ragged shape of the outer contour for the case of the small Λ shows that the potential is nearly zero; near-zero values are encountered not just along a curve, but in an area, due to numerical imprecision.

13.2 A leaky system of two aquifers

We consider the case of leakage between two confined aquifers separated by a leaky layer of resistance c. The aquifers are numbered from the top down and have transmissivities T_1 and T_2, where the transmissivity T_j for aquifer j is defined in terms of the hydraulic conductivity k_j and the aquifer thickness H_j as

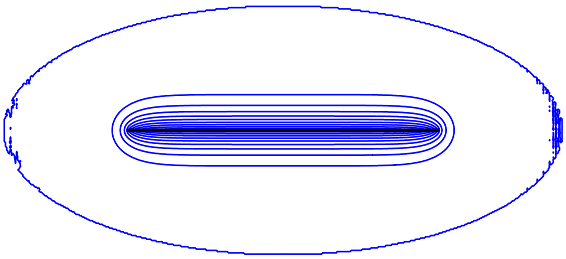


Fig. 5 Contours of constant leakage for $\Lambda = L/10$

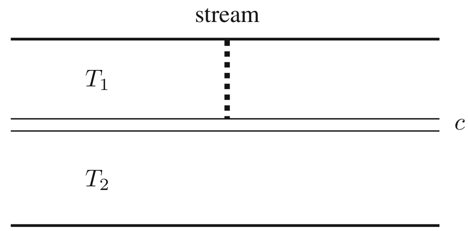


Fig. 6 Section through the aquifer system with the stream

$$T_j = kH_j. \tag{94}$$

We define the total transmissivity T as the sum of the transmissivities of the two aquifers

$$T = T_1 + T_2. \tag{95}$$

We adopt the Dupuit–Forchheimer approximation, i.e., we neglect the resistance to flow in the vertical direction. A section through an aquifer system of this kind, with a stream, is shown in Fig. 6.

We define the discharge potentials in the upper and lower aquifers as follows, see [4, pp. 178–186].

$$\overset{1}{\Phi} = T_1\overset{1}{\phi}, \quad \overset{2}{\Phi} = T_2\overset{2}{\phi}, \tag{96}$$

where $\overset{1}{\phi}$ and $\overset{2}{\phi}$ are the heads in aquifers 1 and 2, respectively. We introduce a comprehensive potential Φ [20] as

$$\Phi = \overset{1}{\Phi} + \overset{2}{\Phi} \tag{97}$$

and a leakage potential G as

$$G = \frac{T_1 T_2}{T}\left(\overset{2}{\phi} - \overset{1}{\phi}\right) = \frac{T_1}{T}\overset{2}{\Phi} - \frac{T_2}{T}\overset{1}{\Phi}. \tag{98}$$

We note that the leakage potential is proportional to the leakage γ [L/T] through the leaky layer, which is positive for upward leakage and equal to the difference in head across the leaky layer divided by the resistance c [T], i.e,

$$\gamma = \frac{\overset{2}{\phi} - \overset{1}{\phi}}{c} = \frac{G}{\Lambda^2}, \tag{99}$$

where Λ is the leakage factor, defined as

$$\Lambda^2 = \frac{cT_1T_2}{T}. \tag{100}$$

The components of the discharge vectors in the upper and lower aquifer are equal to the gradients of the discharge potentials. We introduce the complex discharge functions $\overset{j}{W}$ in aquifer j as

$$\overset{j}{W} = \overset{j}{Q}_x - \mathrm{i}\overset{j}{Q}_y, \tag{101}$$

so that

$$\overset{j}{W} = -2\frac{\partial\overset{j}{\Phi}}{\partial Z}. \tag{102}$$

We consider the case that there is no infiltration from rainfall into the system. Thus, the divergence of the comprehensive discharge vector is zero, and the comprehensive potential is harmonic,

$$\frac{\partial^2\Phi}{\partial Z \partial \bar{Z}} = 0. \tag{103}$$

There is leakage equal to an amount G/Λ^2 out of aquifer 2. Thus, the potential in aquifer 2 satisfies the differential equation (compare (6)),

$$\nabla^2 \overset{2}{\Phi} = 4\frac{\partial^2 \overset{2}{\Phi}}{\partial z \partial \bar{z}} = \frac{16}{L^2}\frac{\partial^2 \overset{2}{\Phi}}{\partial Z \partial \bar{Z}} = \gamma = \frac{G}{\Lambda^2}. \tag{104}$$

We observe from (97) and (98) that we may write $\overset{2}{\Phi}$ as

$$\overset{2}{\Phi} = \frac{T_2}{T}\Phi + G. \tag{105}$$

We substitute this expression for $\overset{2}{\Phi}$ in (104) and obtain, noting that the comprehensive potential is harmonic

$$\frac{\partial^2 G}{\partial Z \partial \bar{Z}} = \frac{L^2}{16}\frac{G}{\Lambda^2}. \tag{106}$$

13.3 A stream in the upper aquifer

We consider the case of an aquifer system that is infinite in extent, with flow from infinity toward a stream in the upper aquifer. The comprehensive potential will be composed of a constant and the potential for a line-sink of some polynomial strength $\sigma(Z)$, expressed in terms of the dimensionless local complex variable Z. We represent this potential as

$$\Phi = \Re\mathfrak{e}\Omega = \Re\mathfrak{e}\sigma(Z)\underset{\mathrm{ls}}{\Omega}(Z). \tag{107}$$

There is no singularity in the lower aquifer, and therefore the singular terms of the functions $T_2/T\,\Phi$ and T_1T_2/T^2G in (105) must cancel. This implies that the strength (i.e., the jump in the normal component of flow, equal to $\sigma(X)$ along the element) of the generating analytic element must be equal to σ_G with

$$\sigma_G = -\frac{T}{T_1}\sigma. \tag{108}$$

The unknowns in the solution that remain, after applying (108), are the coefficients in the polynomial representation of σ, which is

$$\sigma = \sum_{m=0}^{M} a_m Z^m. \tag{109}$$

The unknown real coefficients a_m are computed by requiring that the head along the stream is equal to a constant. The remaining constant, the one in the comprehensive potential, is obtained by requiring that the head at some large distance from the element(s) is given and fixed.

We first consider the case of two high-order elements (order 16) that make an angle of 90°. The first element runs from (−100,0) to (100,0) and the second one from (100,0) to (100,200). The head along both elements is specified at 20 m and the heads far away in both aquifers are 100 m. The hydraulic conductivities and thicknesses in the two aquifers are equal. The hydraulic conductivities are 10 m/day and the thicknesses are 10 m. The resistance of the leaky layer is four days. The leakage factor therefore is

$$\Lambda = \sqrt{\frac{cT_1T_2}{T}} = \sqrt{\frac{4 \times 100^2}{100 + 100}} = \sqrt{2}00 \approx 14.14\,\mathrm{m}; \tag{110}$$

the ratio of $\Lambda/(2L)$ is thus, 0.07, i.e., Λ is 7% of the length of the elements. We use 40 terms both in the far-field expansion of the holomorphic functions and 40 terms in the expansion of functions, i.e., $N = 40$. The final terms in the expansion are so small that the solution does no longer change, i.e., the accuracy falls within machine precision.

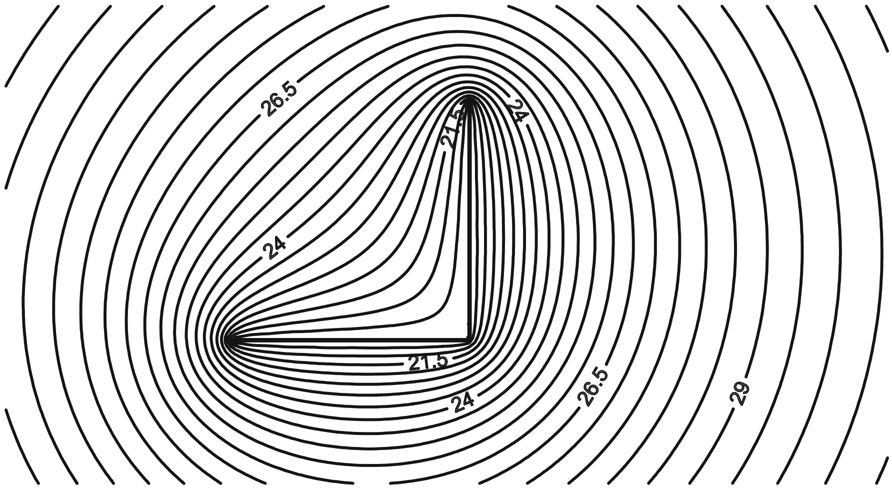


Fig. 7 Heads in the upper aquifer; contour interval 0.5 m. The 20 m contour coincides with the line segments

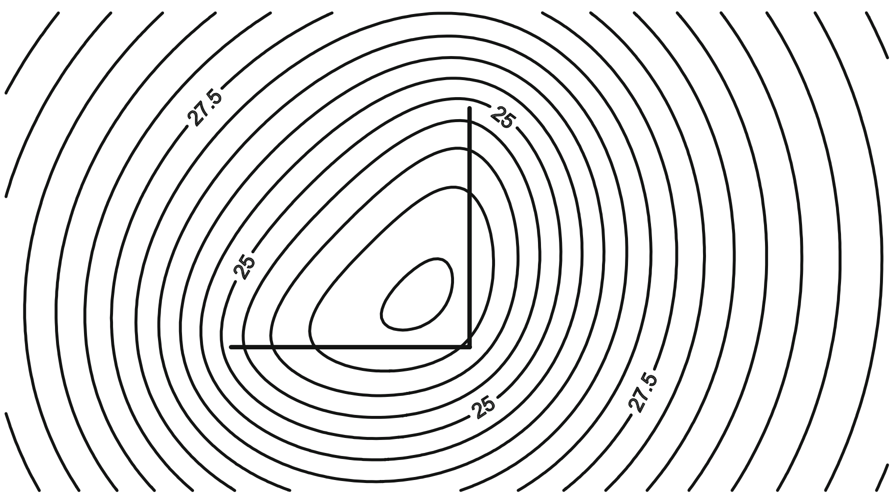


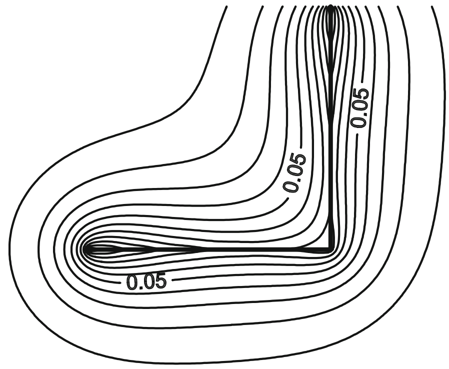
Fig. 8 Heads in the lower aquifer; contour interval 0.5 m

This occurrence can be detected by the program, and the expansion can be broken off once the addition of terms has no effect on the solution.

Plots of the piezometric heads are given in Fig. 7 for the upper and in Fig. 8 for lower aquifer. We present contours of constant leakage in a separate figure, Fig. 9; note that the leakage varies along the elements as it is driven by the clearly variable difference in head; it is largest at the endpoints and at the corner where the elements meet.

The final example is concerned with a river modeled with lower order elements (order 5), but otherwise the same data as for the other examples. The contours of constant head are shown in Fig. 10, the contours of constant head in the lower aquifer in Fig. 11, and the contours of constant leakage in Fig. 12. The leakage varies between zero (the outermost contour corresponds to a leakage of 0.01 m/d), and 0.16 m/d.

Fig. 9 Contours of constant leakage (detail); contour interval 0.01 m/day



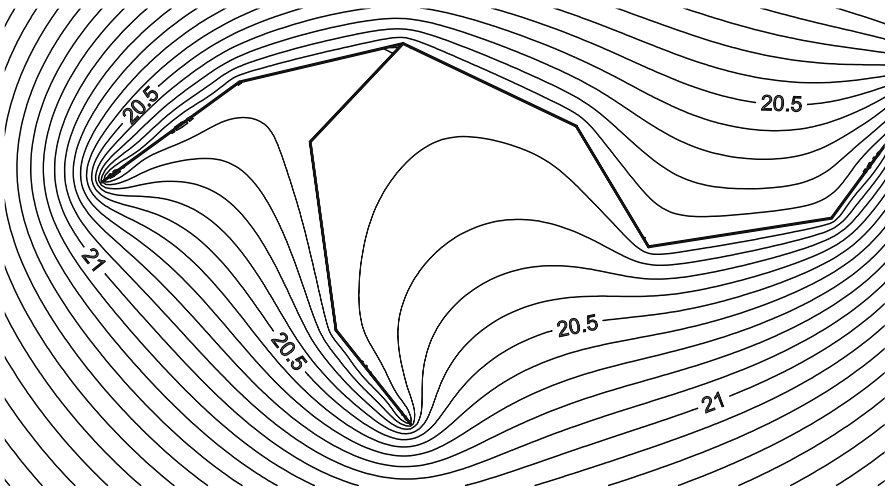


Fig. 10 Piezometric contours in the upper aquifer; the head along the streams is 20 m and the contour interval is 0.1 m

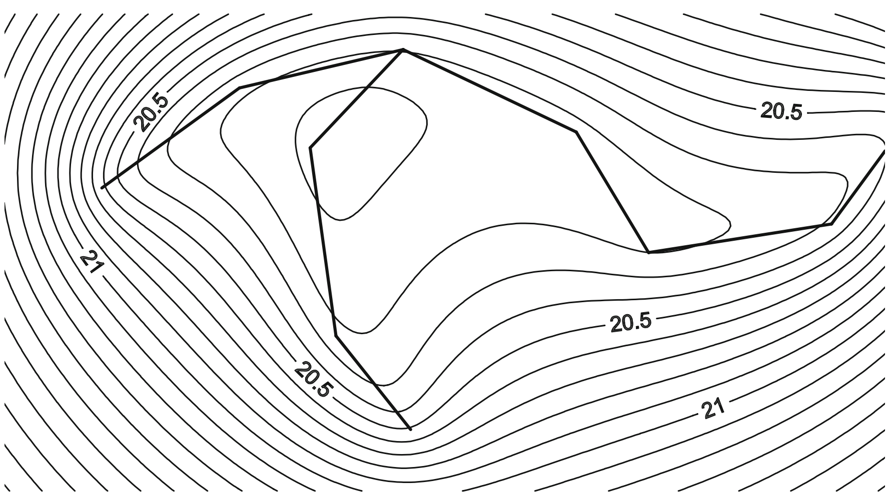
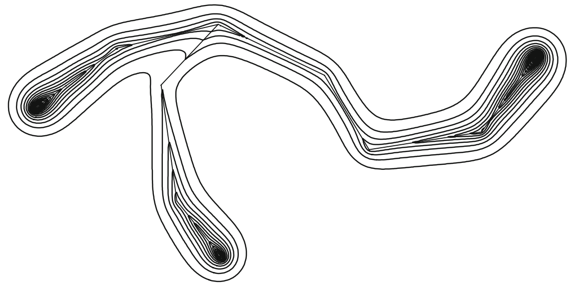


Fig. 11 Piezometric contours in the lower aquifer; note that the elements are not in the lower, but in the upper aquifer. The lowest head is along the shortest closed contour and is 20.1 m

Fig. 12 Contours of constant leakage; the levels vary from 0 to 0.16 m/day and the contour interval is 0.01 m/day. The highest levels are at the endpoints



14 Concluding remarks

This paper is concerned with a new technique suitable for solving boundary-value problems that involve the operator ∇^{2n} where n is a positive integer. The technique is a combination of existing methods: the analytic element method, Wirtinger calculus, and the Adomian decomposition method.

The analytic element method is based upon the idea that boundary-value problems can be solved by superposition of suitably chosen base functions, which each contain certain degrees of freedom. Most base functions are the mathematical representation of discontinuities of either the normal or the tangential component of the vector field along a linear boundary segment. The base functions are then superimposed and their degrees of freedom determined so as to model linear two-sided boundaries, boundaries of sub-domains, or boundaries that separate parts of the domain from other parts.

Analytic elements, at least those existing at present, are all associated with a line or a curve and are dealt with in isolation in terms of derivation, and derive their ultimate effect from their combined effect. Consider, for example, the function

$$\Omega = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} \log \frac{z - z_{j+1}}{z - z_j}, \tag{111}$$

where z_j, $j = 1, \ldots, n$ represents the coordinates of the corner points of a closed polygon and where $z_{n+1} = z_1$. This function consists of the sum of n individual analytic elements, that each create their own field. Upon superposition, however, the resulting function is a constant everywhere, but jumps from the constant value zero to the constant value one, valid inside the polygon. This kind of behavior is not only elegant from a mathematical viewpoint, but has the advantage of straight forward numerical implementation in an object-oriented framework; each analytic element is an object. Such an object-oriented framework is relatively straightforward in terms of design, because the analytic elements do not inherit from one another, but can be viewed as existing side by side, rather than in a hierarchical manner.

Wirtinger calculus has the advantage of compactness of notation and manipulation; using Wirtinger calculus, vectors of two components in two-dimensional space can now be represented by a single complex number, and tensors by half the number of components that would be necessary in Cartesian coordinates. The extension to general two-dimensional problems from ones that are governed by Laplace's equation is only one benefit of Wirtinger calculus. The other one, is that the Laplacian becomes integrable because the $(z, \bar{z})$ complex space is based on using the imaginary characteristics of Laplace's equation as non-Cartesian coordinates.

The decomposition method makes use of subsequent iteration of seeding functions; application of this method to integrate the Laplacian of a function F, for example, would mean that a seeding expression be chosen for $-\partial^2 F/\partial x^2$ and then integrated twice with respect to y to obtain the next term in the expansion. Using Wirtinger calculus, this process is greatly simplified; now singular line elements that are solutions to Laplace's equation can be used as seeding functions.

The combination of these three methods renders a technique, the GAEA, that makes it possible to solve problems that would be very difficult to deal with otherwise. The GAEA is presented in terms of groundwater flow, but is by no means restricted to that field. In fact, other suitable fields include diffusion and heat flow, elasto-dynamics, and poro-elasticity to mention just a few examples. The basic functions presented in this paper can be used for all of the latter applications; effort expended on the optimization of the computational algorithm to evaluate these functions, and on further improvements, such as the superblock approach, (see [21]) will benefit a multitude of applications.

The primary objective of this paper is to present the method, to indicate its applicability to numerous fields, to demonstrate that it works, and to encourage further development by presenting the base functions necessary for further development.

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Appendix A: Determination of the coefficients $\underset{n}{B}_j$

We need to verify that it is possible to choose the constants $\underset{n}{B}_j$ in (71) in such a way that the functions $\underset{n}{\Xi}_m$ satisfy the following condition:

$$\frac{\partial^2 \underset{n}{\Xi}_m}{\partial Z \partial \bar{Z}} = \underset{n-1}{\Xi}_m. \tag{112}$$

We recall the form of $\underset{n}{\Xi}_m$ as

$$\underset{n}{\Xi}_m = \sum_{j=0}^{n-1} \frac{\underset{n}{B}_j}{(n-j)!} (\bar{Z} - Z)^{n-j} \underset{n+j}{F}{}_m(Z), \tag{113}$$

where the constants $\underset{n}{B}_j$ are real,

$$\mathfrak{Im} \underset{n}{B}_j = 0, \qquad \forall \quad n, j. \tag{114}$$

We differentiate (113) with respect to Z and obtain

$$\frac{\partial \underset{n}{\Xi}_m}{\partial Z} = \sum_{j=0}^{n-1} \frac{-\underset{n}{B}_j}{(n-j-1)!} (\bar{Z} - Z)^{n-j-1} \underset{n+j}{F}{}_m(Z) \quad + \sum_{j=0}^{n-1} \frac{\underset{n}{B}_j}{(n-j)!} (\bar{Z} - Z)^{n-j} \underset{n+j-1}{F}{}_m(Z). \tag{115}$$

We replace j by $j-1$ in the first sum and adjust the limits accordingly

$$\begin{aligned} \frac{\partial \underset{n}{\Xi}_m}{\partial Z} &= \sum_{j=1}^{n-1} \frac{-\underset{n}{B}_{j-1}}{(n-j)!} (\bar{Z} - Z)^{n-j} \underset{n+j-1}{F}{}_m(Z) \\ &\quad - \underset{n}{B}_{n-1} \underset{2n-1}{F}{}_m(Z) + \sum_{j=0}^{n-1} \frac{\underset{n}{B}_j}{(n-j)!} (\bar{Z} - Z)^{n-j} \underset{n+j-1}{F}{}_m(Z). \end{aligned} \tag{116}$$

The two sums can now be combined

$$\frac{\partial \underset{n}{\Xi}_m}{\partial Z} = \sum_{j=0}^{n-1} \frac{\underset{n}{B}_j - \underset{n}{B}_{j-1}}{(n-j)!} (\bar{Z} - Z)^{n-j} \underset{n+j-1}{F}{}_m(Z) - \underset{n}{B}_{n-1} \underset{2n-1}{F}{}_m(Z), \tag{117}$$

where we define $\underset{n}{B}_{-1}$ as zero,

$$\underset{n}{B}_{-1} = 0. \tag{118}$$

We differentiate (117) with respect to $\bar{Z}$ and obtain

$$\frac{\partial^2 \underset{n}{\Xi}_m}{\partial Z \partial \bar{Z}} = \sum_{j=0}^{n-1} \frac{\underset{n}{B}_j - \underset{n}{B}_{j-1}}{(n-j-1)!} (\bar{Z} - Z)^{n-j-1} \underset{n+j-1}{F}_m(Z). \tag{119}$$

The function $\underset{n-1}{\Xi}_m$ must be equal to $\partial^2 \underset{n}{\Xi}_m / (\partial Z \partial \bar{Z})$, compare (112), so that

$$\sum_{j=0}^{n-1} \frac{\underset{n-1}{B}_j}{(n-j-1)!} (\bar{Z} - Z)^{n-j-1} \underset{n+j-1}{F}_m(Z) = \sum_{j=0}^{n-1} \frac{\underset{n}{B}_j - \underset{n}{B}_{j-1}}{(n-j-1)!} (\bar{Z} - Z)^{n-j-1} \underset{n+j-1}{F}_m(Z). \tag{120}$$

Note that we extended to $n-1$ the sum of terms that represents $\underset{n-1}{\Xi}_m$, which implies that we have defined the constants $\underset{n}{B}_n$ as

$$\underset{n}{B}_n = 0, \qquad n \geq 1. \tag{121}$$

The conditions (120) are indeed satisfied, provided that the constants meet the following constraints

$$\underset{n}{B}_j = \underset{n}{B}_{j-1} + \underset{n-1}{B}_j, \qquad j = 0, 1, 2, \ldots n-1, \qquad n = 1, 2, \ldots. \tag{122}$$

We may choose

$$\underset{0}{B}_0 = 1, \tag{123}$$

so that the first few applications of (122) give

$$\underset{1}{B}_0 = 1, \quad \underset{2}{B}_0 = \underset{2}{B}_{-1} + \underset{1}{B}_0 = \underset{1}{B}_0 = 1, \quad \underset{2}{B}_1 = \underset{2}{B}_0 + \underset{1}{B}_1 = 1.$$

The recursive relation (122) together with (123) makes it possible to determine all of the coefficients in the expression for the potential, with the exception of the coefficients a_m which are to be determined from the boundary conditions along the element. Note that the coefficients can all be computed a priori; they are independent both of the order m and the coefficient β.

Appendix B: Determination of the coefficients $\underset{n}{C}_j$

We demonstrate in this appendix that the coefficients $\underset{n}{C}_j$ in (87) can be chosen such that the following condition is met

$$\frac{\partial^2 \underset{n}{H}_m}{\partial Z \partial \bar{Z}} = \underset{n-1}{H}_m. \tag{124}$$

We differentiate (87) with respect to Z and obtain

$$\begin{aligned} \frac{\partial \underset{n}{H}_m}{\partial Z} &= \sum_{j=0}^{n-1} \frac{-\underset{n}{C}_j}{(n-j-1)!} (\bar{Z} - Z)^{n-j-1} \left[\underset{n+j}{G}_m + (-1)^{n-j} \underset{n+j}{\bar{G}}_m \right] \\ &\quad + \sum_{j=0}^{n} \frac{\underset{n}{C}_j}{(n-j)!} (\bar{Z} - Z)^{n-j} \underset{n+j-1}{G}_m. \end{aligned} \tag{125}$$

We rewrite this equation as follows

$$\begin{aligned} \frac{\partial \underset{n}{H}_m}{\partial Z} &= \sum_{j=0}^{n-2} \frac{-\underset{n}{C}_j}{(n-j-1)!} (\bar{Z} - Z)^{n-j-1} \left[\underset{n+j}{G}_m + (-1)^{n-j} \underset{n+j}{\bar{G}}_m \right] \\ &\quad + \sum_{j=0}^{n-1} \frac{\underset{n}{C}_j}{(n-j)!} (\bar{Z} - Z)^{n-j} \underset{n+j-1}{G}_m - \underset{n}{C}_{n-1} \left[\underset{2n-1}{G}_m - \underset{2n-1}{\bar{G}}_m \right] + \underset{n}{C}_n \underset{2n-1}{G}_m. \end{aligned} \tag{126}$$

The discharge function must be continuous across the line element for all $n \geq 1$. The terms in the two sums vanish along the element, and are thus continuous. The other terms exhibit jumps in their imaginary parts; these parts must cancel and the constants must satisfy the following conditions

$$\underset{n}{C}_n = 2\underset{n}{C}_{n-1}, \quad n = 1, 2, \ldots, \tag{127}$$

so that

$$\frac{\partial \underset{n}{H}_m}{\partial Z} = \sum_{j=0}^{n-2} \frac{-\underset{n}{C}_j}{(n-j-1)!}(\bar{Z}-Z)^{n-j-1}\left[\underset{n+j}{G}_m + (-1)^{n-j}\underset{n+j}{\bar{G}}_m\right]$$
$$+\sum_{j=0}^{n-1} \frac{\underset{n}{C}_j}{(n-j)!}(\bar{Z}-Z)^{n-j}\underset{n+j-1}{G}_m + \underset{n}{C}_{n-1}\left[\underset{2n-1}{G}_m + \underset{2n-1}{\bar{G}}_m\right].$$

We replace j by $j-1$ in the first sum, and obtain

$$\frac{\partial \underset{n}{H}_m}{\partial Z} = \sum_{j=1}^{n-1} \frac{-\underset{n}{C}_{j-1}}{(n-j)!}(\bar{Z}-Z)^{n-j}\left[\underset{n+j-1}{G}_m + (-1)^{n-j+1}\underset{n+j-1}{\bar{G}}_m\right]$$
$$+\sum_{j=0}^{n-1} \frac{\underset{n}{C}_j}{(n-j)!}(\bar{Z}-Z)^{n-j}\underset{n+j-1}{G}_m + \underset{n}{C}_{n-1}\left[\underset{2n-1}{G}_m + \underset{2n-1}{\bar{G}}_m\right].$$

We define

$$\underset{n}{C}_{-1} = 0, \tag{128}$$

so that the lower limit in the first term can be changed from 1 to zero. We can now combine the two sums into a single one

$$\frac{\partial \underset{n}{H}_m}{\partial Z} = \sum_{j=0}^{n-1} \frac{(\bar{Z}-Z)^{n-j}}{(n-j)!}\left[\left(\underset{n}{C}_j - \underset{n}{C}_{j-1}\right)\underset{n+j-1}{G}_m - (-1)^{n-j+1}\underset{n}{C}_{j-1}\underset{n+j-1}{\bar{G}}_m\right]$$
$$+\underset{n}{C}_{n-1}\left[\underset{2n-1}{G}_m + \underset{2n-1}{\bar{G}}_m\right]. \tag{129}$$

We can include the last term in the sum by extending it to $j = n$, using (127) so that

$$\frac{\partial \underset{n}{H}_m}{\partial Z} = \sum_{j=0}^{n} \frac{(\bar{Z}-Z)^{n-j}}{(n-j)!}\left[\left(\underset{n}{C}_j - \underset{n}{C}_{j-1}\right)\underset{n+j-1}{G}_m - (-1)^{n-j+1}\underset{n}{C}_{j-1}\underset{n+j-1}{\bar{G}}_m\right]. \tag{130}$$

We differentiate this expression with respect to $\bar{Z}$ and obtain

$$\frac{\partial^2 \underset{n}{H}_m}{\partial Z\partial\bar{Z}} = \sum_{j=0}^{n-1} \frac{(\bar{Z}-Z)^{n-j-1}}{(n-j-1)!}\left[\left(\underset{n}{C}_j - \underset{n}{C}_{j-1}\right)\underset{n+j-1}{G}_m - (-1)^{n-j+1}\underset{n}{C}_{j-1}\underset{n+j-1}{\bar{G}}_m\right]$$
$$+\sum_{j=0}^{n} \frac{(\bar{Z}-Z)^{n-j}}{(n-j)!}\left[-(-1)^{n-j+1}\underset{n}{C}_{j-1}\underset{n+j-2}{\bar{G}}_m\right].$$

We note that the contribution for $j = 0$ in the second sum is zero, because $\underset{n}{C}_{-1} = 0$, by definition. We replace j by $j+1$ in the second sum, and obtain

$$\frac{\partial^2 \underset{n}{H}_m}{\partial Z\partial\bar{Z}} = \sum_{j=0}^{n-1} \frac{(\bar{Z}-Z)^{n-j-1}}{(n-j-1)!}\left[\left(\underset{n}{C}_j - \underset{n}{C}_{j-1}\right)\underset{n+j-1}{G}_m - (-1)^{n-j+1}\underset{n}{C}_{j-1}\underset{n+j-1}{\bar{G}}_m\right]$$
$$+\sum_{j=0}^{n-1} \frac{(\bar{Z}-Z)^{n-j-1}}{(n-j-1)!}\left[-(-1)^{n-j}\underset{n}{C}_j\underset{n+j-1}{\bar{G}}_m\right].$$

We can combine the two sums into a single one, which gives

$$\frac{\partial^2 \underset{n}{H}_m}{\partial Z \partial \bar{Z}} = \sum_{j=0}^{n-1} \frac{(\bar{Z}-Z)^{n-j-1}}{(n-j-1)!} \left[\left(\underset{n}{C}_j - \underset{n}{C}_{j-1} \right) \underset{n+j-1}{G}_m + (-1)^{n-j+1} (\underset{n}{C}_j - \underset{n}{C}_{j-1}) \underset{n+j-1}{\bar{G}}_m \right] \tag{131}$$

or

$$\frac{\partial^2 \underset{n}{H}_m}{\partial Z \partial \bar{Z}} = \sum_{j=0}^{n-1} \frac{(\bar{Z}-Z)^{n-j-1}}{(n-j-1)!} \left(\underset{n}{C}_j - \underset{n}{C}_{j-1} \right) \left[\underset{n+j-1}{G}_m + (-1)^{n-j-1} \underset{n+j-1}{\bar{G}}_m \right]. \tag{132}$$

Expression (132) must be identical to expression (87) for $n-1$, see (50),

$$\frac{\partial^2 \underset{n}{H}_m}{\partial Z \partial \bar{Z}} = \underset{n-1}{H}_m = \sum_{j=0}^{n-1} \frac{(\bar{Z}-Z)^{n-j-1}}{(n-j-1)!} \underset{n-1}{C}_j \left[\underset{n+j-1}{G}_m + (-1)^{n-j-1} \underset{n+j-1}{\bar{G}}_m \right]. \tag{133}$$

Equations 132 and 133 are identical, provided that the constants $\underset{n}{C}_j$ satisfy the following condition

$$\underset{n}{C}_j = \underset{n}{C}_{j-1} + \underset{n-1}{C}_j, \quad j = 0, 1, 2, \ldots n-1, \quad n = 1, 2, \ldots, \tag{134}$$

where

$$\underset{n}{C}_{-1} = 0 \tag{135}$$

with the additional constraint given by (127)

$$\underset{n}{C}_n = 2\underset{n}{C}_{n-1}, \quad n = 1, 2, \ldots. \tag{136}$$

It follows from (134) and (135) that

$$\underset{n}{C}_0 = \underset{n-1}{C}_0. \tag{137}$$

The constants $\underset{n}{C}_j$ may be computed as follows. First, (137) is applied to compute all constants $\underset{n}{C}_0$, using that $\underset{0}{C}_0 = 1$. Next, $\underset{1}{C}_1$ is evaluated using (136), so that (134) can be used for $n = 2$. Again, (136) is applied to compute $\underset{2}{C}_2$. Proceeding in this manner, alternately applying (134) and (136), all constants $\underset{n}{C}_j$ can be evaluated.

Appendix C: The asymptotic expansion of the holomorphic function G_j

We will expand the functions $\underset{n}{G}_j(Z)$ about $1/Z = 0$ by first expanding $\underset{0}{G}_j(Z)$, and then integrating the result n times with respect to Z to obtain the expansion of $\underset{n}{G}_j(Z)$

$$\underset{0}{G}_j(Z) = \sum_{k=1}^{j} \alpha_{jk} \left\{ \underset{0}{f}_k(Z-1) - (-1)^{j-k} \underset{0}{f}_k(Z+1) \right\} + \underset{n}{P}_j(Z), \tag{138}$$

where we recall (61),

$$\underset{n}{f}_m(Z) = \frac{m!}{(n+m)!} Z^{m+n} \left[\log Z - \sum_{j=1}^{n} \frac{1}{m+j} \right], \quad n \geq 1, \tag{139}$$

$$\underset{n}{f}_m(Z) = \frac{m!}{(n+m)!} Z^{m+n} \log Z, \quad n = 0, \tag{140}$$

which we may express as

$$\underset{1s}{\Omega}_m = \frac{1}{2\pi} \underset{0}{G}_m = \frac{1}{2\pi} \left\{ (Z^m - 1) \log \frac{Z-1}{Z+1} - 2s_m \log(Z+1) + P_m(Z) \right\}, \tag{141}$$

where

$$s_m = 0, \quad m \text{ odd}; \quad s_m = 1, \quad m \text{ even}. \tag{142}$$

We will expand the function $\underset{0}{G}_m$ about infinity as follows

$$\underset{0}{G}_m = (Z^m - 1)\sum_{j=1}^{\infty}\beta_j Z^{-j} - 2s_m \log Z - 2s_m \log\left[1 + \frac{1}{Z}\right] + P_m(Z). \tag{143}$$

The polynomial $P_m(Z)$ is such that it cancels all powers of Z greater than or equal to zero in this expansion. What remains is

$$\underset{0}{G}_m = \sum_{k=m+1}^{\infty}\beta_j Z^{m-k} - \sum_{k=1}^{\infty}\beta_j Z^{-k} - 2s_m \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}Z^{-k} - s_m 2\log Z. \tag{144}$$

We combine terms and introduce new constants $\underset{j}{c}_k$

$$\underset{0}{G}_m = \sum_{k=1}^{\infty}\underset{m}{c}_k Z^{-k} - 2s_m \log Z, \tag{145}$$

where we obtain expressions for the constants $\underset{j}{c}_k$ by requiring that (144) be equal to (145).

The functions $\underset{n}{G}_j$ are defined such that

$$\frac{\mathrm{d}\underset{n}{G}_m}{\mathrm{d}Z} = \underset{n-1}{G}_m. \tag{146}$$

The index j refers to the power of Z in expression (141), and merely affects the values of the constants $\underset{j}{c}_k$. Since the form of the expressions does not depend on j, we may drop this index for the sake of brevity, provided that we remember that the values of the constants in the result of our analysis depend upon the power j. We obtain the far-field expansions of the functions $\underset{n}{G}_j$ by integrating expression (145) n times. We apply this integration term by term, and consider a single term $\underset{n}{h}_m$, defined as

$$\underset{0}{h}_m = Z^{-m} \tag{147}$$

and

$$\frac{\mathrm{d}\underset{n}{h}_m}{\mathrm{d}Z} = \underset{n-1}{h}_m. \tag{148}$$

We obtain, by repeated integration of $\underset{0}{h}_m$

$$\underset{n}{h}_m = \frac{(-1)^n Z^{n-m}}{(m-1)(m-2)\ldots(m-n)} \qquad n < m \tag{149}$$

and for the special case that $n = m$

$$\underset{m}{h}_m = \frac{(-1)^{m-1}}{(m-1)!}\log Z. \tag{150}$$

The expression for $n > m$ is

$$\underset{m+k}{h}_m = \frac{(-1)^{m-1}}{(m-1)!k!}Z^k\left[\log Z - \sum_{j=1}^{k}\frac{1}{j}\right]. \tag{151}$$

We introduce constants $\underset{n}{\alpha}_m$ and $\underset{n}{\beta}_m$ and δ_k as follows

$$\underset{n}{\alpha}_m = \frac{(-1)^n}{(m-1)(m-2)\dots(m-n)}, \quad n < m \tag{152}$$

and

$$\underset{k}{\beta}_m = \frac{(-1)^{m-1}}{(m-1)!k!}. \tag{153}$$

and, finally

$$\delta_k = \sum_{j=1}^{k} \frac{1}{j}. \tag{154}$$

The expansion of $\underset{n}{G}_j$ thus becomes

$$\underset{n}{G}_j = \sum_{m=1}^{\infty} \underset{j}{c}_m \underset{n}{h}_m. \tag{155}$$

We replace the limit ∞ by a finite number M and use the expressions obtained for $\underset{n}{h}_m$, which gives, using expressions (149) and (151) , noting that the first one applies for $m > n$ and the second one for $m \le n$

$$\underset{n}{G}_j = \sum_{m=n+1}^{M} \underset{j}{c}_m \underset{n}{\alpha}_m Z^{-(m-n)} + \sum_{m=1}^{n} \underset{j}{c}_m \underset{n-m}{\beta}_m Z^{n-m}[\log Z - \delta_{n-m}]. \tag{156}$$

We redefine indices to simplify this expression as follows

$$\underset{n}{G}_j = \sum_{k=1}^{M-n} \underset{k+n}{c}{}_j \underset{n}{\alpha}_{n+k} Z^{-k} + \sum_{k=0}^{n-1} \underset{n-k}{c}{}_j \underset{k}{\beta}_{n-k} Z^{k}[\log Z - \delta_k]. \tag{157}$$

Note that the integrations with respect to Z introduce a constant for each value of n. Subsequent integrations will result in the addition of a polynomial. We compute the coefficients in this polynomial by setting each of the expansions of the functions $\underset{n}{G}_j$ plus a constant equal to the corresponding original function, thereby building the polynomial term by term as the process of integration progresses.

Notes on references

Please note: Master's and Ph.D. theses are available from the University of Minnesota Library by logging onto the library website of the University of Minnesota (www.umn.edu) and searching in MNCAT plus. All three theses referred to in this paper show up in the search.

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